**Coherent States**

The coherent states of the HO are the states which behave in a classical fashion. Their expectations of x, p, change with time like we would expect from a classical oscillator.

**Construction of Coherent State**

Without going through all details, these states are formed from a generalization of the displacement operator acting on the ground state. Noting,



Let’s consider the translation operator:



and now we’ll change it a little bit, to accommodate complex ‘displacements’:



Note the operator is still unitary. And if z is real, then we just recover the original translation operator. Then the coherent state is:



where |0> is the HO ground state. Now let’s consider the overlap of such a coherent state with the excited states in general. First, we have:



and so, using the identity eA+B = eAeBe[A,B]/2 = eBeAe[B,A]/2. So,



Last, let’s evince that the coherent state is an eigenstate of the annihilation operator, using that formula (see below)…



So it is, with eigenvalue z. Likewise



So the coherent state is an eigenvector for both. Last thing I need to do is examine the overlap between two states |z1> and |z2>. This is:



So evidentally, there is non-zero overlap between coherent states. Turns out the coherent states, though not orthogonal to each other, do form a complete set, such that:



Another nice property, pertaining to the matrix elements of a normal ordered operator A(a†,a):



**<x> and <p> in a coherent state**

Let’s work out a few expectations. We can use that eA+B = eAeBe[A,B]/2 property.



Now we’ll do the expectation of position:



We can use the expansion from the QM foundations file, that:



Noting,



we have:



and so, when we take the expectation, we get:



What is the expectation of momentum?



Evaluating the translated operator, we have:



Taking the expectation, we find:



What’s the energy expectation?



That seems weird. Is there supposed to be a zero-point energy term there? Yeah, that’s just the difference between <p2> and <p>2 I guess. Can also say:



Well, I think they typically define α = z/√(mω/2). Then we have:



**Coherent state ΔxΔp uncertainty**

Is there uncertainty the minimum? Let’s see. We’ve already found,



And reviewing our calculation of <H>, we see,



which translates to:



So let’s form the uncertainty,



So,



When we restore units, this would be ℏ/2. So it is the minimum uncertainty packet.

**Time-dependence of a coherent state and how <x(t)>, <p(t)> follow classical expectations**

What is the time dependence of the wavefunction?



Now we’ll note that the time dependence of a function of an operator, is just the function of the time dependence of the operator. So,



And so, we find



So we just have another time-dependent coherent state. And we’ll note that it evolves in time harmonically, according to classical expectations [just take Re and Im parts of argument and you’ll see]. For instance consider how <x> evolves.



and this is how a classical oscillator behaves. For consider:



In accordance with N2L. Also,



So it’s got the classical initial conditions right too.

**Example**

A particle in a harmonic well is prepared in a minimum uncertainty packet, with initial position x0 = 0, and initial momentum p0 = p. What is the most probable energy it will be found to have, upon measurement? What is the probability it will be in the ground state?

Going to restore units.



And we have:



So,



The energy overlaps are:



For what n is this max? Let’s take ln of both sides,



Now take derivative,



Filling in z,



Well this makes sense, as its just the state corresponding to the given initial energy. Probability of being in ground state is:

